Section 7.7: Approximate Integration

Objective: In this lesson, you learn

☐ How to find approximate values of definite integrals using the Midpoint Rule and the Trapezoidal Rule and obtain the error bounds involved.

I. Midpoint Rule and Trapezoidal Rule.

Suppose we are to find approximate values of definite integrals that are difficult to evaluate. Since the definite integral $\int_a^b f(x)\,dx$ is defined as a limit of Riemann sums, any Riemann sum could be used as an approximation to the integral.

If we divide $[a, b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$, then we have

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x,$$

$a = x_0$ $\quad x_1 \quad x_2 \quad \cdots \quad x_{n-2} \quad x_{n-1} \quad x_n = b$

where x_i^* is any point in the i^{th} subinterval $[x_{i-1}, x_i]$.

- If x_i^* is chosen to be $x_i^* = x_{i-1}$, the left endpoint of the interval, then

  $[x_0, x_1], [x_1, x_2], [x_2, x_3] \cdots$

  $[x_{n-1}, x_n]$

  $$\int_a^b f(x)\,dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x,$$

  which is called **the left-endpoint approximation**.

- If x_i^* is chosen to be $x_i^* = x_i$, the right endpoint of the interval, then

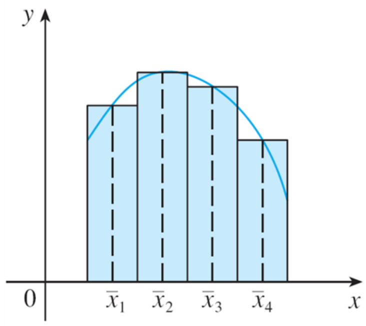
  $$\int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x,$$

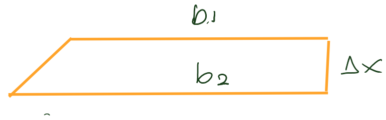
  which is called **the right-endpoint approximation.**

- If x_i^* is chosen to be $x_i^* = \bar{x}_i = \frac{x_i + x_{i-1}}{2}$, the midpoint of the interval, then

  $$\int_a^b f(x)\,dx \approx M_n = \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots f(\bar{x}_n)\right],$$

  which is called **the midpoint approximation**.





$\Delta x = \frac{b-a}{n}$

II. Trapezoidal Rule.

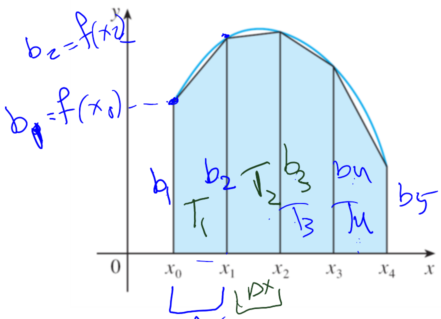
the area of T_1

$= \text{height}(\text{base } 1 + \text{base } 2)$

$= \frac{\Delta x}{2}(b_1 + b_2)$

the are of T_2

$= \Delta x (b_2 + b_3)$



the Area under the curve is

$= \frac{\Delta x}{2}(b_1+b_2) + \frac{\Delta x}{2}(b_2+b_3) + \frac{\Delta x}{2}(b_3+b_4) + \frac{\Delta x}{2}(b_4+b_5)$

$= \frac{\Delta x}{2}(b_1 + 2b_2 + 2b_3 + 2b_4 + b_5) = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right)$

The **Trapezoidal Rule**, results from averaging the left endpoint approximation and right endpoint approximation as follows:

$$\int_a^b f(x)\,dx = \frac{1}{2}(L_n + R_n) \approx \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\,\Delta x + \sum_{i=1}^{n} f(x_i)\,\Delta x\right]$$

$$= \frac{\Delta x}{2}\left[\sum_{i=1}^{n}\left(f(x_{i-1}) + f(x_i)\right)\right]$$

$$= \frac{\Delta x}{2}\left[\left(f(x_0) + f(x_1)\right) + \left(f(x_1) + f(x_2)\right) + \cdots \left(f(x_{n-1}) + f(x_n)\right)\right]$$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

The reason for the name Trapezoidal Rule is that the area of the trapezoid whose area on each interval represents the average of the left-and right-endpoint approximations is

$$\frac{\Delta x}{2}\left[f(x_{i-1}) + f(x_i)\right].$$

Error Bounds for Midpoint Rule and Trapezoidal Rule.

The error bounds of the Midpoint Rule and Trapezoidal Rule are as follows. (The proof can be found in books on numerical analysis.)

Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint rules, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2}.$$

*

Example 1: Consider $\int_1^2 \frac{1}{x}\,dx = \ln x \Big|_1^2 = \ln(2) - \cancel{\ln(1)}^{0} = \ln(2) = 0.6931\ldots$

a. Use the Midpoint and Trapezoidal Rule with $n = 4$ to approximate the given integral.

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = 1/4$$

$1 = \frac{4}{4}$, $5/4$, $\frac{6}{4}$, $\frac{7}{4}$, $\frac{8}{4} = 2$

$$\left[\overset{x_0}{\tfrac{4}{4}}, \overset{x_1}{\tfrac{5}{4}}\right], \left[\tfrac{5}{4}, \overset{x_2}{\tfrac{6}{4}}\right], \left[\tfrac{6}{4}, \overset{x_3}{\tfrac{7}{4}}\right], \left[\tfrac{7}{4}, \overset{x_4}{2}\right]$$

Midpoint $\frac{\frac{4}{4}+\frac{5}{4}}{2} = \frac{9}{8}$, $\frac{\frac{5}{4}+\frac{6}{4}}{2} = \frac{11}{8}$, $\frac{13}{8}$, $\frac{15}{8}$

$$M_4 = \Delta x \left(f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4) \right)$$
$$= \frac{1}{4}\left(f\left(\tfrac{9}{8}\right) + f\left(\tfrac{11}{8}\right) + f\left(\tfrac{13}{8}\right) + f\left(\tfrac{15}{8}\right) \right)$$
$$= \frac{1}{4}\left(\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15} \right) = 0.69121989$$

under estimate

Trapezoidal

$$T_4 = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4) \right]$$
$$= \frac{\Delta x}{2}\left(f(1) + 2f\left(\tfrac{5}{4}\right) + 2f\left(\tfrac{6}{4}\right) + 2f\left(\tfrac{7}{4}\right) + f(2) \right)$$
$$= \frac{1}{8}\left(1 + 2\cdot\frac{4}{5} + 2\cdot\frac{4}{6} + 2\cdot\frac{4}{7} + \frac{1}{2} \right) = 0.6970238$$

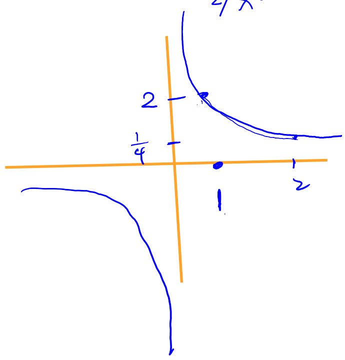
over estimate

b. Give an error bound involved in these approximations.

$f(x) = \frac{1}{x} = x^{-1}, \quad f'(x) = -x^{-2}, \quad f''(x) = +2x^{-3}$

$1 \le x \le 2$

$f''(x) = \frac{2}{x^3} \Rightarrow |f''(x)| \le 2 \rightarrow K$



$$|E_T| \le \frac{K(b-a)^3}{12n^2} = \frac{\cancel{2}(2-1)^3}{6\,\cancel{12}\cdot(4)^2} = \frac{1}{96} = 0.01041666$$

$$|E_M| \le \frac{K(b-a)^3}{24n^2} = \frac{2(2-1)^3}{24(4)^2} = 0.005208$$

c. How large should n be to guarantee that the both Rules approximations for the integral are accurate to within 0.0001?

ⓐ Midpoint $|E_M| \le \frac{K(b-a)^3}{24n^2} \le 0.0001$

$K=2,\ b=2,\ a=1$

$$\frac{\cancel{2}(2-1)^3}{12\,\cancel{24}\,n^2} \le 0.0001$$

$$0.0001 * 12n^2 \ge 1 \Rightarrow n^2 \ge \frac{1}{0.0001*12} = 833.333$$

$$n \ge \sqrt{833.333} = 28.86 =$$

at least $\boxed{n \ge 29}$

ⓑ Trap. $|E_T| \le \frac{K(b-a)^3}{12n^2} \le 0.0001$

$$\frac{\cancel{2}(2-1)^3}{\cancel{12}\,6n^2} \le 0.0001$$

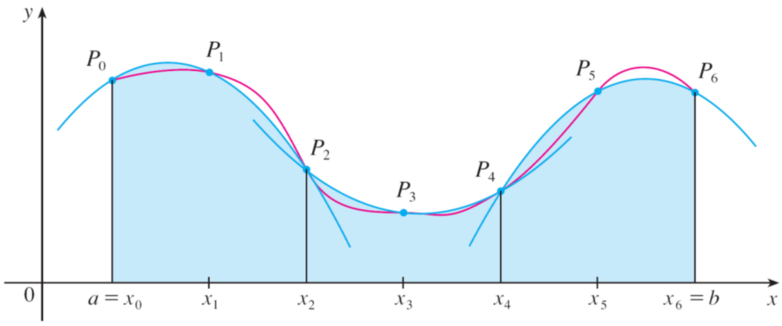
$$0.0001 * 6n^2 \ge 1 \Rightarrow n^2 \ge \frac{1}{0.0001*6} = 1666.666$$

$$n \ge 40.824$$

at least $n \ge 41$

III. Simpson's Rule.

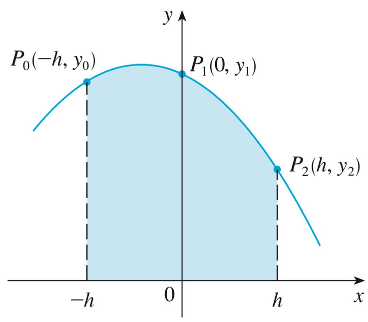
Another rule for approximate integration results from using parabolas rather than straight line segments to approximate a curve. Divide $[a, b]$ into n subintervals of equal length $h = \Delta x = \frac{b-a}{n}$ and assume that n is an **even** number. Then on each consecutive pair of intervals, we approximate the curve $y = f(x) \geq 0$ by a parabola. A typical parabola passes through three consecutive points $P_i(x_i, y_i)$, $P_{i+1}(x_{i+1}, y_{i+1})$, and $P_{i+2}(x_{i+2}, y_{i+2})$, where $y_k = f(x_k)$, for $k = i$, $i+1$, and $i+2$.



For simplification, consider the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$.

The equation of the parabola through P_0, P_1, and P_2 is of the form $y = Ax^2 + Bx + C$.

So the area under the parabola from $x = -h$ to $x = h$ is



$$\int_{-h}^{h} \left(Ax^2 + Bx + C\right) dx = 2\int_0^h \left(Ax^2 + C\right) dx = 2\left[A\frac{x^3}{3} + Cx\right]_0^h = 2\left(A\frac{h^3}{3} + Ch\right) = \frac{h}{3}\left(2Ah^2 + 6C\right)$$

But since the parabola passes through $P_0(-h, y_0)$, $P_1(0, y_1)$, and $P_2(h, y_2)$, we have

$$\begin{aligned} y_0 &= A(-h)^2 + B(-h) + C = Ah^2 - Bh + C, \\ y_1 &= C, \\ y_2 &= Ah^2 + Bh + C. \end{aligned}$$

$Ah^2 - \cancel{Bh} + C + 4C + Ah^2 \cancel{+Bh} + C$
$= 2Ah^2 + 6C$

So $2Ah^2 + 6C = y_0 + 4y_1 + y_2$.

Thus,

$$\int_{-h}^{h} \left(Ax^2 + Bx + C\right) dx = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right).$$

Shifting this parabola horizontally will not change the area under it. That is, the parabola through P_0, P_1, and P_2 from $x = x_0$ to $x = x_2$ is still such that its area is

$$\frac{h}{3}\left(y_0 + 4y_1 + y_2\right).$$

Similarly, the area under the parabola through P_2, P_3, and P_4 from $x = x_2$ to $x = x_4$ is

$$\frac{h}{3}\left(y_2 + 4y_3 + y_4\right).$$

So adding the areas under all the parabolas yields

$$\begin{aligned}\int_a^b f\left(x\right) dx &\approx \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) + \frac{h}{3}\left(y_2 + 4y_3 + y_4\right) + \cdots + \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right) \\ &= \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right).\end{aligned}$$

Although we have derived this approximation for the case in which $f\left(x\right) \geq 0$, it is a reasonable approximation for any continuous function f.

We state this result called **Simpson's Rule**:

Simpson's Rule

$\int_a^b f\left(x\right) dx \approx$

$$S_n = \frac{\Delta x}{3}\left[f\left(x_0\right) + 4f\left(x_1\right) + 2f\left(x_2\right) + 4f\left(x_3\right) + \cdots + 2f\left(x_{n-2}\right) + 4f\left(x_{n-1}\right) + f\left(x_n\right)\right],$$

where n is even and $\Delta x = (b-a)/n$.

Error Bounds for Simpson's Rule:

We also state without proof the error bound for Simpson's Rule.

Suppose $\left|f^{(4)}\left(x\right)\right| \leq K$ for $a \leq x \leq b$. If E_S is the error involved in using Simpson's Rule, then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

Example 2: Consider $\int_1^2 \frac{1}{x}\, dx$

$\Delta x = \frac{2-1}{4} = \frac{1}{4}$

a. Use Simpson's Rule with $n = 4$ to approximate the given integral.
b. Give an error bound involved in this approximation.
c. How large should n be to guarantee that the approximation for the integral is accurate to within 0.0001?

$\left[\underset{x_0}{\frac{4}{4}}, \underset{x_1}{\frac{5}{4}}\right], \left[\frac{5}{4}, \underset{x_2}{\frac{6}{4}}\right], \left[\frac{6}{4}, \underset{x_3}{\frac{7}{4}}\right], \left[\frac{7}{4}, \underset{x_4}{\frac{8}{4}}\right].$

a)

$$S_4 = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$$

$$= \frac{1}{12}\left(f\left(\frac{4}{4}\right) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{6}{4}\right) + 4f\left(\frac{7}{4}\right) + f\left(\frac{8}{4}\right)\right)$$

$$= \frac{1}{12}\left(1 + \frac{16}{5} + \frac{8}{6} + \frac{16}{7} + \frac{4}{8}\right)$$

$$\simeq 0.693539683$$

b) $f(x) = x^{-1} \to f'(x) = -x^{-2} \to f''(x) = 2x^{-3} \to f'''(x) = -6x^{-4} \to f''''(x) = 24x^{-5}$

$|f^{(4)}(x)| \leq 24 \to K.$

$$|E_S| \leq \frac{K(b-a)^5}{180 n^4} = \frac{24(2-1)^5}{180(4)^4} = 0.0005208$$

c) Find n such that

$$\frac{K(b-a)^5}{180 n^4} \leq 0.0001$$

$$\frac{24(2-1)^5}{180 n^4} \leq 0.0001 \Rightarrow 24 \leq 0.0001 * 180 n^4$$

$$\Rightarrow n^4 \geq \frac{24}{0.0001 * 180} = 1333.333$$

$$n \geq 6.0427$$

$n \geq 8$ n is even